Study Material

B.Sc. I (Hons)

Paper : 1.

Topic : Solution of Cubic Equation by Cardan's method.

Material Sl. no. — 1.

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Solution of Cubic Equation by Cardan's method.

Reduction of Cubic Equation into standard form

Let a general cubic equation be

$$a_0x^3+3a_1x^2+3a_2x+a_3=0 \quad \text{---- (1)}$$

Let us substitute $x=y+h$ in equation (1).

So (1) takes the form

$$a_0(y+h)^3+3a_1(y+h)^2+3a_2(y+h)+a_3=0$$

or, $a_0y^3+3ha_0y^2+3a_0h^2y+a_0h^3+3a_1y^2+6a_1hy+3a_1h^2+3a_2y+3a_2h+a_3=0$

or, $a_0y^3+(3ha_0+3a_1)y^2+(3a_0h^2+6a_1h+3a_2)y+(a_0h^3+3a_1h^2+3a_2h+a_3)=0$ — (ii)

Now we like to omit the second term in the cubic. So the constant $3ha_0+3a_1=0$

$$\Rightarrow h=-\frac{3a_1}{3a_0} \quad \text{or,} \quad h=-\frac{a_1}{a_0}$$

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Now. $3(a_0h^2 + 2a_1h + a_2) = 3\cdot\left[a_0\cdot\left(-\frac{a_1}{a_0}\right)^2 + 2a_1\left(-\frac{a_1}{a_0}\right) + a_2\right]$

$$= 3\left[\frac{a_1^2}{a_0} - \frac{2a_1^2}{a_0} + a_2\right]$$

$$= 3\cdot\left[\frac{a_0a_2 - a_1^2}{a_0}\right]$$

$$= \frac{3H}{a_0}, \text{ where } H = a_0a_2 - a_1^2$$

Again. $(a_0h^3 + 3a_1h^2 + 3a_2h + a_3)$

$$= a_0\left(-\frac{a_1}{a_0}\right)^3 + 3a_1\left(-\frac{a_1}{a_0}\right)^2 - 3a_2\frac{a_1}{a_0} + a_3$$

$$= -\frac{a_1^3}{a_0^2} + \frac{3a_1^3}{a_0^2} - \frac{3a_1a_2}{a_0} + a_3$$

$$= \frac{-a_1^3 + 3a_1^3 - 3a_0a_1a_2 + a_3a_0^2}{a_0^2}$$

$$= \frac{2a_1^3 - 3a_0a_1a_2 + a_0^2a_3}{a_0^2}$$

$$= \frac{G}{a_0^2}, \text{ where } G = 2a_1^3 - 3a_0a_1a_2 + a_0^2a_3$$

Putting these values in equation (II) we get

$$a_0y^3 + \frac{3H}{a_0}y + \frac{G}{a_0^2} = 0$$

$$\text{or, } y^3 + \frac{3H}{a_0^2}y + \frac{G}{a_0^3} = 0 \quad \text{—— (III)}$$

$\boxed{\text{Note:}}$ If α, β, γ be the roots of the equation (I), then clearly $\alpha - h, \beta - h, \gamma - h$ ~~are the~~ ~~ro~~ i.e $\alpha + a_1/a_0,\ \beta + a_1/a_0,\ \gamma + a_1/a_0$ are the roots of (iii).

As α, β, γ are the roots of (i) so from the relation between roots and co-efficients we have

$$\alpha+\beta+\gamma = -\frac{3a_1}{a_0} \quad \text{or,} \quad \frac{a_1}{a_0} = -\frac{1}{3}(\alpha+\beta+\gamma)$$

So roots of (iii) are given by

$$\alpha - \frac{1}{3}(\alpha+\beta+\gamma), \quad \beta - \frac{1}{3}(\alpha+\beta+\gamma), \quad \gamma - \frac{1}{3}(\alpha+\beta+\gamma)$$

$$\text{Or, } \frac{1}{3}(2\alpha-\beta-\gamma), \quad \frac{1}{3}(2\beta-\alpha-\gamma), \quad \frac{1}{3}(2\gamma-\alpha-\beta)$$

Now we multiply the roots of (iii) by a_0.

Let α_1 be any of its roots.

Let $z = a_0\alpha_1$ or, $\alpha_1 = \frac{z}{a_0}$

As α_1 is a root of (iii) so

$$\alpha_1^3 + \frac{3H}{a_0}\alpha_1 + \frac{G}{a_0^3} = 0$$

$$\text{Or, } \frac{z^3}{a_0^3} + \frac{3H}{a_0}\cdot\frac{z}{a_0} + \frac{G}{a_0^3} = 0$$

$$\text{or, } \boxed{z^3 + 3Hz + G = 0}$$

Which is the standard form of a Cubic.