Study Material

B.SC. II

Sub: Mathematics

Paper: III

Topic: Integral Calculus (Beta and Gamma Functions)

Gamma and Beta Functions

Some Important Relation.

1. $\int_0^\infty e^{-at} t^{x-1} dt = \frac{\Gamma(x)}{a^x}$, $x>0$ and for any $a>0$.

Soln: $\int_0^\infty e^{-at} t^{x-1} dt$

Let $at = v$

or, $dt = \frac{dv}{a}$

t	0	∞
v	0	∞

$= \int_0^\infty e^{-v} \left(\frac{v}{a}\right)^{x-1} \frac{dv}{a}$

$= \frac{1}{a^x} \int_0^\infty e^{-v} v^{x-1} dv$

$= \frac{1}{a^x} \Gamma(x)$.

2. $\Gamma(x+1) = x\Gamma(x)$, $x>0$.

Soln:

We know that

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt = \int_\varepsilon^A e^{-t} t^{x-1} dt$$

Where $\varepsilon \to 0^+$ & $A \to \infty$

$$= \left[e^{-t} \frac{t^x}{x} \right]_\varepsilon^A + \frac{1}{x} \int_\varepsilon^A e^{-t} t^x dt$$

As $\varepsilon \to 0+$ and $A \to \infty$.

$\left[e^{-t} \frac{t^x}{x} \right]_\varepsilon^A$ vanishes

Therefore $\int_0^\infty e^{-t} t^{x-1} dt = \frac{1}{x} \int_0^\infty e^{-t} t^x dt$

or, $\Gamma(x) = \frac{1}{x} \int_0^\infty e^{-t} t^{(x+1)-1} dt$

or, $\Gamma(x) = \frac{1}{x} \Gamma(x+1)$

or, $x\Gamma(x) = \Gamma(x+1)$

or, $\Gamma(x+1) = x\Gamma(x)$.

Corollary:

$$\Gamma(x+n) = (x+n-1)(x+n-2)\cdots(x+1)\Gamma(x),$$

for all integers $n \geq 1$ and for all $x > 0$.

3. $\Gamma(1) = \underline{1}$.

Soln: We know that

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad x > 0$$

So $\Gamma(1) = \int_0^\infty e^{-t} t^{1-1} dt$

$$= \int_0^\infty e^{-t} dt$$

$$= -\left[e^{-t}\right]_0^\infty$$

$$= -0 + \underline{1}$$

$$= \underline{1}.$$

4. $\Gamma(n+1) = n!$, n being positive integer

Soln: $\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots$

$$= n \cdot n-1 \cdots 3 \cdot 2 \cdot 1\,\Gamma(1) = n!$$