Study Material
B.Sc. II
Sub: Mathematics
Paper: III
Topic: Integral Calculus (Beta and Gamma functions)

Gamma and Beta Functions

Gamma function.

Gamma function is defined as

$$\int_0^\infty e^{-t} t^{x-1} dt, \quad x > 0$$

and is denoted as $\Gamma(x)$.

i.e $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$

Theorem

$\Gamma(x)$ is continuous and differential function for $x > 0$.

Proof Let $x_0 \in (0, \infty)$ be an arbitrary point.

Let a and b such that $0 < a < x < b$

Now we observe the ~~contin~~ convergence of $\Gamma(x)$ on the interval $a \le x \le b$.

We have $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$.

$$= \int_0^1 e^{-t} t^{x-1} dt + \int_1^\infty e^{-t} t^{x-1} dt$$

$$= \Gamma_1(x) + \Gamma_2(x) \quad (\text{say})$$

Where $\Gamma_1(x) = \int_0^1 e^{-t} t^{x-1} dt$ and

$$\Gamma_2(x) = \int_1^\infty e^{-t} t^{x-1} dt$$

Now $|e^{-t} t^{x-1}| \le t^{a-1} = M_1(t)$, $a \le x \le b$

$\boxed{\forall\ 0 \le t \le 1}$

and $\int_0^1 M_1(t)\, dt$ converge

Hence $\Gamma_1(x)$ converges uniformly for $a \le x \le b$ and $0 \le t \le 1$

Again $|e^{-t} t^{x-1}| \le e^{-t} t^{b-1} = M_2(t)$ (say)

, $a \le x \le b$ & for $t \ge 1$.

and $\int_1^{\infty} M_2(t)\,dt$ converges.

Hence $\Gamma_2(x)$ converges uniformly for $a \le x \le b$ and $t \ge 1$

Since $\Gamma_1(x)$ and $\Gamma_2(x)$ ~~con~~. and consequently their sum $\Gamma(x)$ are all continuous in $a \le x \le b$.

So in particular $\Gamma(x)$ is continuous at x_0, $x_0 \in (0, \infty)$

Since x_0 is an arbitrary no. so $\Gamma(x)$ is continuous for all $x > 0$

Similarly we can show that $\Gamma(x)$ is differentiable.

Hence the theorem.