Material Sl. no ~~43~~ 45

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Curvature

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Evolute

The Locus of the centre of curvature of a given curve is called evolute and the given curve is called involute.

Properties.

The normal at any point of the given curve is the tangent to the evolute to the centre of curvature for that particular point.

Example.

Let ρ_1 and ρ_2 be the ~~a~~ radius of curvature at the ~~co-ordinal~~ corresponding points of the curve $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ and its evolute, Then show that $\rho_1^2 + \rho_2^2 =$ Constant.

Soln Equation of given curve is

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$x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$

differentiating above we get :

$$\frac{dx}{d\theta} = a(1 - \cos\theta)$$

& , $$\frac{dy}{d\theta} = a\sin\theta$$

So, $$\frac{dy}{dx} = \frac{a\sin\theta}{a(1-\cos\theta)}$$

$$= \frac{2\sin\theta/2 \cos\theta/2}{2\sin^2\theta/2}$$

$$= \cot\theta/2$$

Again differentiating above we get

$$\frac{d^2y}{dx^2} = -\operatorname{cosec}^2\theta/2 \left(\tfrac{1}{2}\right) \cdot \frac{d\theta}{dx}$$

$$= \frac{-\frac{1}{2}\operatorname{cosec}^2\theta/2}{a(1-\cos\theta)} = -\frac{1}{4a}\operatorname{cosec}^4\theta/2$$

Hence $\rho_1 = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

$$= \frac{(1+\cot^2\theta/2)^{3/2}}{-\frac{1}{4a}\operatorname{cosec}^4\theta/2}$$

$$= -4a\frac{\operatorname{cosec}^3\theta}{\operatorname{cosec}^4\theta/2} = -4a\sin\theta/2$$

Again $\rho_2 = \dfrac{d\rho_1}{d\psi}$

$$= \frac{d\rho_1}{d\theta}\cdot\frac{d\theta}{d\psi} =$$

$$= -4a\cos\theta/2\cdot\frac{1}{2}\cdot\frac{d\theta}{d\psi}$$

$$= -2a\cos\theta/2\,\frac{d\theta}{d\psi}$$

Now we have $y_1 = \cot\theta/2$ or, $\tan\psi = \cot\theta/2$

differentiating we get

$$\sec^2\psi\,\frac{d\psi}{d\theta} = -\frac{1}{2}\operatorname{cosec}^2\theta/2$$

or, $$\frac{d\psi}{d\theta} = -\frac{1}{2}\frac{\operatorname{cosec}^2\theta/2}{1+\tan^2\psi}$$

$$= -\frac{1}{2}\frac{\operatorname{cosec}^2\theta/2}{\operatorname{cosec}^2\theta/2}$$

$$= -1/2$$

So, $\rho_2 = -2a\cos\theta/2\cdot(-2) = 4a\cos\theta/2$

Hence $\rho_1^2 + \rho_2^2 = 16a^2\sin^2\theta/2 + 16a^2\cos^2\theta/2$

$$= 16a^2(\sin^2\theta/2 + \cos^2\theta/2) = 16a^2 = \text{Constant.}$$