Material Sl. no. 35

Pedal Equation

Problem

1. Show that the pedal equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ w.r.t. its focus is

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1.$$

Soln.

Shifting the origin to the focus of the ellipse, the equation of the ellipse becomes

$$\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{——— (1)}$$

Parametric form of the ellipse is

$$x = a\cos\theta - ae$$

and $y = b\sin\theta$.

So, $\frac{dx}{d\theta} = -a\sin\theta \;\Big|\; \frac{dy}{d\theta} = b\cos\theta.$

So, $\frac{dy}{dx} = -\frac{b}{a} \cdot \frac{\cos\theta}{\sin\theta} = -\frac{b}{a} \cot\theta$.

Now equation of the tangent at (x, y) is

$$y - b\sin\theta = \left(-\frac{b}{a}\cot\theta\right)\left(x - (a\cos\theta - ae)\right)$$

$$\text{or, } ay - ab\sin\theta = b\cot\theta\left(a\cos\theta - ae - x\right)$$

$$\text{or, } ay - ab\sin\theta = ab\cos\theta\cot\theta - abe\cot\theta - xb\cot\theta$$

$$\text{or, } ay + b\cot\theta . x = ab\sin\theta + ab\cot\theta\cos\theta - abe\cot\theta$$

$$\text{or, } b\cot\theta\, x + ay = ab\sin\theta + ab\cot\theta\cos\theta - abe\cot\theta.$$

Let p be the length of perpendicular from the origin to the tangent.

Then $$p = \frac{ab\sin\theta + ab\cot\theta\cos\theta - abe\cot\theta}{\sqrt{b^2\cot^2\theta + a^2}}$$

$$= \frac{ab\sin\theta + ab\cot\theta\cos\theta - abe\cot\theta}{\sqrt{b\cos^2\theta + a^2\sin^2\theta}} \cdot \sin\theta$$

$$= \frac{ab\sin^2\theta + ab\cos^2\theta - abe\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

$$= \frac{ab - abe\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

or, $\frac{1}{p^2} = \frac{a^2\sin^2\theta + b^2\cos^2\theta}{a^2b^2(1-e\cos\theta)^2}$

We know that $r^2 = x^2 + y^2$

$= (a\cos\theta - ae)^2 + b^2\sin^2\theta$

$= a^2\cos^2\theta - 2a^2e\cos\theta + a^2e^2 + b^2\sin^2\theta$

$= a^2\cos^2\theta - 2a^2e\cos\theta + a^2e^2 + b^2 - b^2\cos^2\theta$

$= (a^2-b^2)\cos^2\theta - 2a^2e\cos\theta + a^2e^2 + b^2$

$= (a^2-b^2)\cos^2\theta - 2a^2e\cos\theta + a^2(1-b^2/a^2) + b^2$

$= (a^2-b^2)\cos^2\theta - 2a^2e\cos\theta + a^2 - b^2 + b^2$

$$= a^2\left(1-\frac{b^2}{a^2}\right)\cos^2\theta - 2a^2 e\cos\theta + a^2$$

$$= a^2e^2\cos^2\theta - 2a^2e\cos\theta + a^2$$

$$= (a - ae\cos\theta)^2$$

$$\therefore\ r = a - ae\cos\theta$$

From ① we have.

$$\frac{1}{p^2} = \frac{a^2(1-\cos^2\theta) + b^2\cos^2\theta}{a^2b^2(1-e\cos\theta)^2}$$

$$= \frac{a^2 - (a^2-b^2)\cos^2\theta}{a^2b^2\left(1-\frac{a-r}{a}\right)^2}$$

$$= \frac{a^2\left\{1-\left(1-\frac{b^2}{a^2}\right)\cos^2\theta\right\}}{a^2b^2\cdot\frac{r^2}{a^2}}$$

$$= \frac{a^2\{1-e^2\cos^2\theta\}}{b^2r^2}$$

$$= \frac{a^2(1+e\cos\theta)(1-e\cos\theta)}{b^2r^2}$$

$\therefore$

$$= \frac{a^2\left(1+\frac{a-r}{a}\right)\left(1-\frac{a-r}{a}\right)}{b^2 r^2}$$

$$= \frac{a^2\left(\frac{2a-r}{a}\right)\frac{r}{a}}{b^2 r^2}$$

$$= \frac{(2a-r)}{b^2 r}$$

So, $\frac{1}{p^2} = \frac{2a-r}{b^2 r}$

or, $\frac{b^2}{p^2} = \frac{2a-r}{r}$

or, $\frac{b^2}{p^2} = \frac{2a}{r} - 1$