Numerical Analysis.

Using the N-R method, find the real roots of the equation $x^2 - 5x + 2 = 0$ correct to three places of decimals.

Soln: Let $f(x) \equiv x^2 - 5x + 2$

x	$f(x)$
0	2.
1	-2.

So there is a real root of the given eqn between 0 and 1.

$f'(x) = 2x - 5$.

Now by ~~NR~~ N-R method,

$$\alpha_{n+1} = \alpha_n - \frac{f(\alpha_n)}{f'(\alpha_n)}$$

$$= \alpha_n - \frac{\alpha_n^2 - 5\alpha_n + 2}{2\alpha_n - 5}$$

$$= \frac{2\alpha_n^2 - 5\alpha_n - \alpha_n^2 + 5\alpha_n - 2}{2\alpha_n - 5} = \frac{\alpha_n^2 - 2}{2\alpha_n - 5}$$

Let the initial approximation of the root be $x_0 = 0$. [since the root lies in (0,1)].

Then the successive approximations are

$$\alpha_1 = \frac{\alpha_0^2 - 2}{2\alpha_0 - 5} = \frac{2}{5} = 0.4.$$

$$\alpha_2 = \frac{\alpha_1^2 - 2}{2\alpha_1 - 5} = \frac{\frac{4}{25} - 2}{\frac{4}{5} - 5} = \frac{\frac{4-50}{25}}{\frac{4-25}{5}}$$

$$= 0.4381$$

$$\alpha_3 = \frac{\alpha_2^2 - 2}{2\alpha_2 - 5} = \frac{(0.4381)^2 - 2}{2(0.4381) - 5} = 0.4384$$

Thus the desired root = 0.438 [correct upto three ~~place~~ decimal places]

x	$f(x)$
1	-2
2	-4
3	-4
4	-2
5	$+2$.

So the other root lies in (4,5).

Taking initial approximation $x_0 = 4$ and proceeding as before we get,

$\alpha_1 = 4.6667$.

$\alpha_2 = 4.5641$

$\alpha_3 = 4.5615$

$\alpha_4 = 4.5615$

So the other root = 4.562, (correct upto three decimal places).

since the equation is quadratic there is no other root.

2. Use N-R method to deduce the iterative procedure

$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$ for evaluating $\sqrt{a}$ as the solution of the equation $x^2 - a = 0$. Hence compute $\sqrt{2}$, using $x_0 = 1.4$.

Let $x = \sqrt{a}$
$\therefore x^2 - a = 0$
so $\sqrt{a}$ is the root of $x^2 - a = 0$

Soln:

Here $f(x) = x^2 - a$

$\therefore f'(x) = 2x$.

from N-R method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^2 - a}{2x_n}$$

$$= \frac{x_n^2 + a}{2x_n}$$

or, $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$.

Using $x_0 = 1.4$. We get

$$x_1 = \frac{1}{2}(x_0 + a/x_0) = \frac{1}{2}\left(1.4 + \frac{2}{1.4}\right).$$

$$= \frac{3.96}{2.8}$$

$$= 1.414$$

$$x_2 = \frac{1}{2}(x_1 + a/x_1)$$

$$= \frac{1}{2}\left(1.414 + \frac{2}{1.414}\right).$$

$$= 1.41421$$

Thus $\sqrt{2} = 1.414$ [correct upto four significant figure].

Prove that $\sqrt[k]{a}$ can be evaluated by using the iterative procedure

$$x_{n+1} = \frac{1}{k}\left[(k-1)\cdot x_n + \frac{a}{x_n^{k-1}}\right]$$

Hence compute $\sqrt[3]{2}$ using $x_0 = 1.25$.

~~Soln:~~

Soln: Let $x = \sqrt[k]{a}$

$\therefore x^k - a = 0.$

Let $f(x) = x^k - a$,

$\therefore f'(x) = kx^{k-1}$.

N-R method gives,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^k - a}{k x_n^{k-1}}$$

$$= \frac{(k-1)x_n^k + a}{k x_n^{k-1}}$$

$$= \frac{1}{k}\left[(k-1)\cdot x_n + \frac{a}{x_n^{k-1}}\right].$$

Using $x_0 = 1.25$, $a = 2$, $k = 3$. we get,

$$x_1 = \frac{1}{3}\left[(3-1)\cdot x_0 + \frac{2}{x_0^{3-1}}\right]$$

$$= 1.26$$

$x_2 = 1.259921$

$x_3 = 1.259921$.

Hence $\sqrt[3]{2} \approx 1.259921$