For B.Sc. I (Math Hons)
Paper - II

Symmetric Functions of Roots.

A function g involving two or more than two variables is said to be a symmetric function if g remains unaltered when we interchange any two of its variables.

For example, $g(x,y,z) = xy + yz + zx$ is a symmetric functions of x, y, z.

Again if $g(x,y,z) = xy + yz$. then g is not a symmetric function in x, y, z.

A symmetric function of the roots is an expression involving the roots of an equation which remains unaltered if any two of the roots be interchanged.

For example, if α, β, γ be the roots of a cubic equation, then $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$ is a symmetric function of the roots, but $\alpha^3\beta + \beta^3\gamma + \gamma^3\alpha$ is not a symmetric function of the roots.

Worked Example

(1) If α, β, γ be the roots of the cubic equation $x^3 + px^2 + qx + r = 0$, ~~find the~~ calculate the values of the following symmetric function of the roots.

(i) $\Sigma\alpha^2$, (ii) $\Sigma\alpha^2\beta$ (iii) $\Sigma\alpha^2\beta^2$, (iv) $\Sigma\frac{1}{\alpha}$, (v) $\Sigma\frac{1}{\alpha^2}$

(vi) $\Sigma\alpha^3$, (vii) $\Sigma\alpha^3\beta^3$, (viii) $\Sigma\frac{\beta^2+\gamma^2}{\beta\gamma}$ (ix) $\Sigma\frac{\beta^2+\gamma^2}{\beta+\gamma}$

Soln: As α, β, γ are the roots,

$$\Sigma\alpha = -p, \quad \Sigma\alpha\beta = q, \quad \alpha\beta\gamma = -r$$

(i) $\Sigma\alpha^2 = \alpha^2+\beta^2+\gamma^2$

$$= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$$

$$= (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$$

$$= (-p)^2 - 2q$$

$$= p^2 - 2q$$

(ii) $\Sigma\alpha^2\beta = \Sigma\alpha\,\Sigma\alpha\beta - 3\alpha\beta\gamma$

$$= -p\cdot q + 3r = 3r - pq$$

(iii) $\Sigma\alpha^2\beta^2 = \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2$

$$= (\alpha\beta+\beta\gamma+\gamma\alpha)^2 - 2\alpha^2\beta\gamma - 2\alpha\beta^2\gamma - 2\alpha\beta\gamma^2$$

$$= (\Sigma\alpha\beta)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma)$$

$$= (\Sigma\alpha\beta)^2 - 2\alpha\beta\gamma\,\Sigma\alpha$$

$$= q^2 - 2(-r)(-p)$$

$$= q^2 - 2pr$$

(iv) $\Sigma\frac{1}{\alpha} = \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$

$$= \frac{\beta\gamma+\gamma\alpha+\alpha\beta}{\alpha\beta\gamma} = \frac{q}{-r} = -q/r$$

(v) $\sum \frac{1}{\alpha^2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$

$$= \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right)^2 - 2\left(\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha}\right)$$

$$= \left(\sum \frac{1}{\alpha}\right)^2 - 2\left(\frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}\right)$$

$$= \left(\frac{-q}{r}\right)^2 - 2\frac{(-p)}{-r}$$

$$= \frac{q^2}{r^2} - \frac{2p}{r}$$

$$= \frac{q^2 - 2pr}{r^2}$$

(vi) $\sum \alpha^3 = \alpha^3 + \beta^3 + \gamma^3$

$$= (\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2) - \alpha^2\beta - \alpha^2\gamma - \beta^2\alpha - \beta^2\gamma - \gamma^2\alpha - \gamma^2\beta$$

$$= \sum\alpha \sum\alpha^2 - \sum\alpha^2\beta$$

$$= (-p)(p^2 - 2q) - (3r - pq)$$

$$= -p^3 + 2pq - 3r + pq$$

$$= 3pq - p^3 - 3r.$$

(vii), (viii) & (ix) left undone as an exercise to the students.

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