Study Material.

B. Sc. - II (Math Hons)

Paper - 3

Sequence & Series.

Material Sl. no. → I'II

Rajat Subhra Das.
Assistant Professor
Dr. L.K.V.D. College.

Limit of a Sequence

A sequence $\{x_n\}$ is said to have a limit l ($l \in \mathbb{R}$) if for ~~any~~ given any positive number ε (however small) there exists a natural number m (depends on ε) such that $|x_n - l| < \varepsilon$ $\forall\ n \geq m$.

i.e. $l - \varepsilon < x_n < l + \varepsilon$ for all $n \geq m$.

Theorem

A real sequence can have atmost one limit.

Proof:

If possible let a sequence $\{x_n\}$ have two distinct limits say l_1 and l_2, where $l_1 \neq l_2$.

Since $x_n \to l_1$, so corresponding to $\varepsilon/2$ ($\varepsilon > 0$) there exists $m_1 \in \mathbb{N}$ such that

$$|x_n - l_1| < \varepsilon/2 \quad \forall\ n \geq m_1$$

Let $\varepsilon > 0$ be any positive number.

Again since $x_n \to l_2$, so corresponding to $\varepsilon/2$ ($\varepsilon/2 > 0$ is a pre-assigned positive quantity). $\exists$ $m_2 \in \mathbb{N}$ such that

$$|x_n - l_2| < \varepsilon/2$$

Let $m = \max\{m_1, m_2\}$

Then $|x_n - l_1| < \varepsilon/2$ $\forall\ n \geq m$

& $|x_n - l_2| < \varepsilon/2$, $\forall\ n \geq m$.

Now $|l_1 - l_2| = |l_1 - x_n + x_n - l_2|$

$$\leq |l_1 - x_n| + |x_n - l_2|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \forall\, n \geq m$$

This means that $|l_1 - l_2|$ is less than any real ~~positive~~ number.

Therefore the only possibility is that $l = l'$, and this proves the theorem.

$\boxed{\text{Convergent sequence}}$

A sequence $\{x_n\}$ is said to be convergent iff it has a finite limit l as its limit.

$\boxed{\text{Examples}}$

1. The sequence $\left\{\frac{1}{n}\right\}$ converges to 0.

2. The sequence $\left\{\frac{n^2+1}{3n^2+5}\right\}$ converges to $1/3$.

$$\left|\frac{n^2+1}{3n^2+5} - \frac{1}{3}\right| = \left|\frac{3n^2+3-3n^2-5}{9n^2+15}\right| = \left|\frac{-2}{9n^2+15}\right|$$

$$= \frac{2}{9n^2+15} < \varepsilon$$

When ever $9n^2+15 > 2/\varepsilon$

i.e $n > \sqrt{\frac{2}{9\varepsilon} - 5/3}$

Let $m = \left[\sqrt{\frac{2}{9\varepsilon} - 5/3}\right]$

So when $n > m$, $\left|\frac{n^2+1}{3n^2+5} - \frac{1}{3}\right| < \varepsilon$ $\therefore \frac{n^2+1}{3n^2+5} \to 1/3$.

Null sequence

A sequence $\{x_n\}$ is said to be null sequence if
$$\lim_{n\to\infty} x_n = 0$$

Example

$\left\{\frac{1}{n^p}\right\}$ is a null sequence. (where $p > 0$)

Theorem

A convergent sequence is bounded.

Proof.

Let $\{x_n\}$ be a convergent sequence and converges to l.
i.e for given any $\varepsilon > 0$, $\exists$ a positive integer (natural no.) m such that
$$|x_n - l| < \varepsilon, \quad \forall\, n \geq m.$$

Now
$$|x_n| = |x_n - l + l|$$
$$\leq |x_n - l| + |l|$$
$$< \varepsilon + |l| \quad \forall\, n \geq m$$

Let $M = \max\{|x_1|, |x_2|, \ldots, |x_m|, \varepsilon + |l|\}$

Then $|x_n| < M \quad \forall\, n$

Hence the theorem.

Rajat Subhra Das
Assistant Professor
ABC LKVD College Tajpur