Study Material

B.Sc. II (Math)

Paper — 4

Topic: Some Important Problems of Laplace Transformation.

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Apply Laplace Transformation method to solve the following D.E.

1. $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+5y=e^{-t}\sin t, \quad y(0)=0,\ y'(0)=1$

$\boxed{\text{Soln}}$ Given differential equation can be written as

$$y''+2y'+5y=e^{-t}\sin t$$

Applying Laplace Transformation we get

$$L(y'')+2L(y')+5L(y)=L(e^{-t}\sin t)$$

or, $s^2L(y)-sy(0)-y'(0)+2sL(y)-2y(0)+5L(y)$
$$=L(e^{-t}\sin t)$$

or, $s^2L(y)-1+2sL(y)+5L(y)=L(e^{-t}\sin t)$

or, $(s^2+2s+5)L(y)-1=L(e^{-t}\sin t)$

We know that, $L(\sin t)=\frac{1}{s^2+1^2}=\frac{1}{s^2+1}$

$\therefore\ L(e^{-t}\sin t)=\frac{1}{(s+1)^2+1}=\frac{1}{s^2+2s+2}$

or, $(s^2+2s+5)\,L(y) = 1 + \frac{1}{s^2+2s+2}$

or, $\cancel{s^2}\cdot L(y) = \frac{\cancel{s^3+2s^2+2s+1}\,(s^2+2s+3)}{(s^2+2s+2)(s^2+2s+5)}$

or, $L(y) = \dfrac{(s+1)^2+2}{((s+1)^2+1)((s+1)^2+4)}$

or, $y = L^{-1}\left(\dfrac{(s+1)^2+2}{((s+1)^2+1)((s+1)^2+4)}\right)$

or, $y = e^{-t}L^{-1}\left(\dfrac{s^2+2}{(s^2+1)(s^2+4)}\right)$

$= \cancel{e^{-t}L^{-1}\left(\dfrac{1}{s^2+4} + \dfrac{1}{(s^2+1)(s^2+4)}\right)}$

$= \dfrac{e^{-t}}{2}L^{-1}\left(\dfrac{2}{s^2+4} + \dfrac{2}{(s^2+1)(s^2+4)}\right)$

$= \dfrac{e^{-t}}{2}L^{-1}\left(\dfrac{2}{s^2+2^2}\right) + \dfrac{e^{-t}}{2}L^{-1}\left(\dfrac{2}{(s^2+1)(s^2+4)}\right)$

$= \dfrac{e^{-t}}{2}\sin 2t + \dfrac{e^{-t}}{2}L^{-1}\left(\dfrac{2}{(s^2+1)(s^2+4)}\right)$ —— (1)

Now $L^{-1}\left(\dfrac{1}{s^2+1}\right) = \sin t = f(t)$ say

$L^{-1}\left(\dfrac{2}{s^2+4}\right) = \sin 2t = g(t)$ say

Therefore by convolution theorem we have

$$L^{-1}\left(\frac{2}{(s^2+1)(s^2+4)}\right) = \int_0^t g(x)\, f(t-x)\, dx$$

$$= \int_0^t \sin 2x \sin(t-x)\, dx$$

or $= \int_0^t \sin 2x \left[\sin t \cos x - \cos t \sin x\right] dx$

$= \int_0^t \sin 2x \sin t \cos x \, dx - \int_0^t \sin 2x \cos t \sin x \, dx$

$= 2\sin t \int_0^t \sin x \cos^2 x \, dx - 2\cos t \int_0^t \sin^2 x \cos x \, dx$

For the first integral

let $\cos x = z$

so $-\sin x \, dx = dz$

x	0	t
z	1	$\cos t$

For the second integral

let $\sin x = p$

or, $\cos x \, dx = dp$

x	0	t
p	0	$\sin t$

$= -2\sin t \int_1^{\cos t} z^2 dz - 2\cos t \int_0^{\sin t} p^2 dp$

$= -2\sin t \left[\frac{z^3}{3}\right]_1^{\cos t} - 2\cos t \left[\frac{p^3}{3}\right]_0^{\sin t}$

$= -\frac{2}{3}\left[\sin t \cos^3 t - \sin t + \cos t \sin^3 t\right]$

$= -\frac{2}{3}\left[\sin t \cos t - \sin t\right]$

So from ① we have,

$y = \frac{e^{-t}}{2}\left[\sin 2t - \frac{2}{3}\sin t (\cos t - 1)\right]$