Study Material.

B.Sc. I (Math)

Paper — 1

Topic — Matrices.

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Some Important Problems of Matrices.

Q: State and prove Cayley-Hamilton theorem.

$\boxed{\text{Soln}}$ Cayley-Hamilton theorem:

Statement: Every square matrix satisfies its characteristic equation.

Proof: Let A be a square matrix of order n. Then its characteristic ~~polynomial~~ equation is $|A - xI| = 0$.

Let us suppose that it be

$$b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n = 0.$$

So according to this theorem we have to prove that

$$b_0 I + b_1 A + b_2 A^2 + b_3 A^3 + \cdots + b_n A^n = 0.$$

Now Adj $(A - xI)$ is polynomial in $x$ and of degree atmost $(n-1)$.

Let $\text{Adj}(A - xI) = B_0 + B_1 x + B_2 x^2 + \cdots + B_{n-1} x^{n-1}$.

Now, $(A - xI)\,\text{Adj}(A - xI) = |A - xI|\, I$

$$\Rightarrow (A - xI)(B_0 + B_1 x + B_2 x^2 + \cdots + B_{n-1} x^{n-1})$$

$= |A - \lambda I| I$

Or, $(A - x I)(B_0 + B_1 x + B_2 x^2 + \cdots + B_{n-1} x^{n-1})$

$= (b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n) I.$

Or, $(A - x I)(B_0 + B_1 x + B_2 x^2 + \cdots + B_{n-1} x^{n-1})$

$= b_0 I + b_1 x I + b_2 x^2 I + \cdots + b_n x^n I$

Comparing the co-efficients of like powers of $x$ we get,

$$AB_0 = b_0 I$$
$$AB_1 - B_0 = b_1 I.$$
$$AB_2 - B_1 = b_2 I$$
$$\cdots\cdots$$
$$\cdots\cdots$$
$$AB_{n-1} - B_{n-2} = b_{n-1} I.$$
$$-B_{n-1} = b_n I$$

Pre multiplying by $I, A, A^2, \cdots, A^n$ in the above set of equations respectively we get

$b_0 I + b_1 A + b_2 A^2 + b_3 A^3 + \cdots + b_n A^n$

$= AB_0 + A(AB_1 - B_0) + A^2(AB_2 - B_1) + \cdots$

$+ (-B_{n-1}) A^n$

Or, $b_0 I + b_1 A + b_2 A^2 + b_3 A^3 + \cdots + b_n A^n = 0$

This completes the theorem.

Verify the Cayley-Hamilton theorem for the following matrix.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$ and hence find $A^{-1}$.

Soln: Characteristic equation of the matrix is

$$\begin{vmatrix} 2-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

or, $(\lambda-1)(\lambda-2) = 0$

or, $\lambda^2 - 3\lambda + 2 = 0$.

We have to prove that

$$A^2 - 3A + 2I = 0$$

Now $A^2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

So, $A^2 - 3A + 2I$.

$= \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} - 3\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

Hence Cayley-Hamilton theorem is verified.

We have, $A^2 - 3A + 2I = 0$.

or, $A^2 - 3A = -2I$.

Premultiplying $A^{-1}$ on the both the sides of the above equation we get.

$$A - 3I = -2A^{-1}$$

or, $-2A^{-1} = A - 3I$.

or, $A^{-1} = -\frac{1}{2}(A - 3I)$

or, $$A^{-1} = -\frac{1}{2}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= -\frac{1}{2}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right)$$

$$= -\frac{1}{2}\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$$