Study Material

B.Sc. I (Math)

Paper - I.

Topic : Matrices.

Sub Topic: Linear Equations

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Linear Equations

Let a system of m homogeneous equation in n variables be given by

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = 0$$

$$\cdots \quad \cdots \quad \cdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = 0$$

Where a_{ij}'s is scalars and x_i's are unknowns.

We can write the above system of homogeneous equations in the form

$$AX = O$$

Where $A = [a_{ij}]_{m\times n}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$ & $O = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m\times 1}$

$A = [a_{ij}]$ is called the co-efficient matrix

$X = 0$ is a trivial solution

Now it is easy to show that if X_1 and X_2 are two solutions of the above system of linear equations, then their linear combination is also a solution of the system of equations.

If X_1 is a solution then $AX_1 = 0$

Again if X_2 is a solution then $AX_2 = 0$

Let P_1 and P_2 be any two scalars

$$\text{Now } A(P_1X_1 + P_2X_2) = AP_1X_1 + AP_2X_2$$
$$= P_1(AX_1) + P_2(AX_2)$$
$$= 0$$

This shows that $P_1X_1 + P_2X_2$ is also a solution. Consequently linear combination of any two solution is also a solution.

Theorem :

The number of linearly independent solution of the system of equations $AX = 0$ is $n - r$, where $r = \text{rank}(A)$.

Proof :

As $\text{rank}(A) = r$, so the matrix A has r linearly independent column. Without loss of generality let us suppose that the first r columns are linearly independent

Let us write A as

$$A = [\underbrace{C_1, C_2, \cdots, C_r}_{\text{independent columns}}, C_{r+1}, \cdots C_n]$$

Then the system of equations takes the form

$x_1 C_1 + x_2 C_2 + x_3 C_3 + \cdots + x_r C_r + x_{r+1} C_{r+1} + \cdots + x_n C_n = 0$ —— (1)

Since columns $C_{r+1}, \cdots C_n$ can be written as a linear combination of r columns $C_1, C_2, \cdots, C_r$

So, let $C_{r+1} = K_{11} C_1 + K_{12} C_2 + \cdots + K_{1r} C_r$

$C_{r+2} = K_{21} C_1 + K_{22} C_2 + \cdots + K_{2r} C_r$

$\cdots \quad \cdots$

$\cdots \quad \cdots$

$C_n = K_{p1} C_1 + K_{p2} C_2 + \cdots + K_{pr} C_r$

Where $p = n - r$

The above equations can be written as:

$$\left.\begin{aligned}
&K_{11} C_1 + K_{12} C_2 + \cdots + K_{1r} C_r + (-1) C_{r+1} + 0 \cdot C_{r+2} + \cdots + 0 \cdot C_n = 0 \\
&K_{21} C_1 + K_{22} C_2 + \cdots + K_{2r} C_r + (0) C_{r+1} + (-1) C_{r+2} + \cdots + 0 \cdot C_n = 0 \\
&\cdots \quad \cdots \\
&\cdots \quad \cdots \\
&K_{p1} C_1 + K_{p2} C_2 + \cdots + K_{pr} C_r + 0 \cdot C_{r+1} + 0 \cdot C_{r+2} + \cdots + (-1) C_n = 0
\end{aligned}\right\} \quad (2)$$

So from equation (1) & (2) we have.

$$X_1 = \begin{bmatrix} K_{11} \\ K_{12} \\ \vdots \\ K_{1r} \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} K_{21} \\ K_{22} \\ \vdots \\ K_{2r} \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots, \quad X_p = \begin{bmatrix} K_{p1} \\ K_{p2} \\ \vdots \\ K_{pr} \\ 0 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix}$$

are $P = (n-r)$ solution of the give ~~equat~~ system of equations $AX = 0$

To show the independency of the $(n-r)$ solutions. Let

$$q_1 X_1 + q_2 X_2 + \cdots + q_p X_p = 0$$

So comparing $(r+1)$th, $(r+2)$th, ... nth component we get $-q_1 = 0, -q_2 = 0, \cdots, q_p = 0$

i.e $q_1 = 0, q_2 = 0, \cdots, q_p = 0$

This shows that $X_1, X_2, \cdots, X_p$ are linearly independent solutions of the system of equations $AX = 0$ $(P = n-r)$

To show that any solution of $AX = 0$, can be expressed as a linear combination of $X_1, X_2, \cdots X_{n-r}$ let us suppose that $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ be a solution of $AX = 0$.

Now $X + x_{r+1} X_1 + x_{r+2} X_2 + \cdots + x_n X_{n-r}$ — (3)

is linear combination of the solutions $X_1, X_2, \cdots X_{n-r}$ & X.

As linear combination of solutions of $AX = 0$ is also a solution so $X + x_{r+1} X_1 + x_{r+2} X_2 + \cdots x_n X_{n-r}$ is also a solution.

The last $(n-r)$ components of (3) are zero, so let the remaining component be $z_1, z_2, \cdots z_r$

since a vector with component $z_1, z_2, \cdots z_r, 0, \cdots 0$ is solution.

So from (1) we have.

$$z_1 C_1 + z_2 C_2 + \cdots + z_r C_r = 0.$$

As $C_1, C_2, \cdots C_r$ are linearly independent

So $z_1 = 0 = z_2 = z_3 = \cdots = z_r$

Thus (3) is a zero vector.

Therefore (3) can be written as

$$X = -x_{r+1} X_1 - x_{r+2} X_2 - \cdots - x_n X_{n-r}$$

Thus every solutions of $AX = 0$ can be expressed as a linear combination of $(n-r)$ linearly independent solns.