Study Material

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B. Sc. II (Math)

Topic: Orthogonal Trajectories

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Orthogonal Trajectories.

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Any curve which cuts all member of a family of curves at a fixed angle say α is called a α-trajectory of the family of curves.

If in particular $\alpha = 90°$, then the trajectory is called orthogonal trajectory.

$\boxed{\text{Def}}$

If two family of curves are there and all member of either family cuts all member of the other family at angle $90°$ i.e at right angles, then one family is said to be ~~family~~ orthogonal trajectories to another ~~trajectories~~ family of curves and vice versa.

How to find the orthogonal trajectory of a family of curves

In cartesian co-ordinate

To find the orthogonal trajectories of a family of curve in cartesian co-ordinate system we follow the following steps.

1. Given family of curves is $f(x, y, \alpha) = 0$, where α is a parameter.

2. We differentiate $f(x, y, \alpha) = 0$ w.r.t. x.

   Then we eliminate α.

   Eliminating α we get a differential equation, let it of the form

$$\phi\left(x, y, \frac{dy}{dx}\right) = 0.$$

3. Now replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in the above differential equation.

   So differential equation takes the form

$$\phi\left(x, y, -\frac{dx}{dy}\right) = 0$$

4. We now solve the above differential equation.

   Solution will give the orthogonal trajectories of the given family of curves.

Workedout Example

Find the orthogonal trajectories of the curve $y = ax$, where a is the variable parameters.

$\boxed{\text{Soln.}}$

$$y = ax$$

or, $$\frac{y}{x} = a$$

Differentiating w.r.t. x, we get

or, $$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = 0$$

or, $$\frac{dy}{dx} - \frac{y}{x} = 0$$

For orthogonal trajectory we replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

So replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ we get

$$-\frac{dx}{dy} - \frac{y}{x} = 0$$

or, $$\frac{dx}{dy} = -y/x$$

Or, $$x\,dx = -y\,dy$$

Integrating we get

$$\int x\,dx = -\int y\,dy$$

or, $\frac{x^2}{2} = -y^2/2 + K$

or, $x^2 + y^2 = C$, where $C = 2K$

which is the required orthogonal trajectories.

2. Find the orthogonal trajectories of the following curves.

$y^2 = 4ax$, where a is the parameter.

$\boxed{\text{Soln:}}$ $y^2 = 4ax$.

or, $\frac{y^2}{x} = 4a$

Differential w.r.t. x we get

or, $\frac{2y}{x}\frac{dy}{dx} - \frac{1}{x^2}y^2 = 0$

or, $2\frac{dy}{dx} - \frac{y}{x} = 0$

For orthogonal trajectories we replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, replacing we get

or, $-2\frac{dx}{dy} - \frac{y}{x} = 0$

or, $-2x\,dx = y\,dy$

Integrating we get

$K - x^2 = y^2/2$

or, $2x^2 + y^2 = C$, where $2K = C$.