Study Material.

B.Sc. II (Math)

Topic: Orthogonal Trajectories.

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Orthogonal Trajectories.

Working rule to find orthogonal Trajectories for polar co-ordinates:

To find the orthogonal trajectories of a family of curves in polar co-ordinates we follow the following steps.

1. Given ~~differential~~ equation is
$f(r, \theta, \alpha) = 0$, α is a parameter.

2. We differentiate the above family of equations w.r.t. θ and then eleminate α.
Let after elimination the differential equation takes the form

$$\phi\left(r, \theta, \frac{dr}{d\theta}\right) = 0$$

3. Next we replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$. Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ the above differential equation takes the form

$$\phi\left(r, \theta, -r^2\frac{d\theta}{dr}\right) = 0.$$

4. Now we solve the above differential equation.

5. Solution will give the family of orthogonal trajectories.

Worked out Example.

Find the orthogonal trajectories of the $r = a\theta$.

Soln:

$$r = a\theta$$

Or, $$\frac{r}{\theta} = a$$

Differentiating w.r.t. θ we get

$$\frac{dr}{d\theta}\cdot\frac{1}{\theta} - \frac{1}{\theta^2}r = 0$$

Or, $$\frac{dr}{d\theta} = r/\theta$$

For orthogonal trajectories we replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$.

Replacing we get

$$-r^2\frac{d\theta}{dr} = r/\theta$$

or, $-\theta d\theta = \frac{dr}{r}$.

or, $-\theta^2/2 = +\log r + K$

or, $-\theta^2 = 2\log r + c'$ where $c' = 2K$

or, $\log C + \log e^{-\theta^2} = \log r^2$ where $\log C = -c'$

or, $C \cdot e^{-\theta^2} = r^2$

or, $r^2 = Ce^{-\theta^2}$

Find the orthogonal trajectories of $r^n \sin n\theta = a^n$ where a is a parameter.

Soln! Given $r^n \sin n\theta = a^n$

Taking log on both sides we get

$$\log(r^n \sin n\theta) = \log a^n$$

or, $n\log r + \log\sin n\theta = n\log a$

differentiating w.r.t. θ, we get

$$\frac{n}{r}\cdot\frac{dr}{d\theta} + \frac{n\cos n\theta}{\sin n\theta} = 0$$

or, $\frac{1}{r}\frac{dr}{d\theta} = -\cot n\theta$.

For orthogonal trajectory we replace

$\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$.

Replacing we get

$$-r\frac{d\theta}{dr} = -\cot n\theta$$

or, $\tan n\theta\, d\theta = \frac{dr}{r}$.

Integrating we get

$$\log r = \frac{1}{n}\log(\sec n\theta) + \log c.$$

or, $n\log r = \log(\sec n\theta) + n\log c$.

or, $r^n = c^n \sec n\theta$.

Which is the required orthogonal trajectory.

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