B.Sc. II (Math (Hons))

Study Material

Paper – 4

Topic : Laplace Transformation

Material Sl. no – 3

Prepared by

Rajat Subhra Das.

Assistant Professor

Dr. L.K.V.D. College, Tajpur

Email: srajatdas10@gmail.com

Laplace Transformation

Piecewise continuous function

A function $f(t)$ is said to be piecewise continuous on the interval $[a,b]$ if

i. There exists a partition p of $[a,b]$, ~~i.e for finite~~ ~~no of subintervals~~ $p: a = t_0 < t_1 < t_2 \cdots < t_n = b$ into finite subintervals such that f is continuous on each subintervals (t_{i-1}, t_i), $i = 1, 2, \ldots n$

ii. f has a ~~finite limit~~ jump discontinuity at each t_i, $i = 0, 1, 2, \ldots n$.

Functions of Exponential order

A function $f(t)$ is said to be of exponential order if there exists real number M and a constant a such that

$$|f(t)| < Me^{at}, \quad t \geq t_0.$$

Existence theorem

If f is a piecewise continuous function in ~~every~~ every finite interval in the domain $[0, \infty)$ and of exponential order i.e $|f(t)| < Me^{at}, t \geq t_0$, then the laplace transformation of f exists for all $p \geq a$.

Proof:
$$L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^{t_0} e^{-st} f(t)\, dt + \int_{t_0}^\infty e^{-st} f(t)\, dt$$

$$= I + I_1$$

As f is piecewise continuous function, so I exists.

Now $\left|\int_{t_0}^{\infty} e^{-st} f(t)\,dt\right| \le \int_{t_0}^{\infty} |e^{-st} f(t)|\,dt$

$$\le M \int_{t_0}^{\infty} e^{-st} \cdot e^{at}\,dt$$

$$\le M \int_0^{\infty} e^{-(s-a)t}\,dt$$

$$= M \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty}$$

$$= M\left[0 + \frac{1}{+(s-a)}\right]$$

$$= \frac{M}{s-a}, \quad s > a$$

$$\therefore \left|\int_{t_0}^{\infty} e^{-st} f(t)\,dt\right| \le \frac{M}{s-a}.$$

So, I_2 exists if $s > a$.

Consequently I_1 and I_2 i.e. the integral $\int_0^{\infty} e^{-st} f(t)\,dt$ exists, $(s > a)$

Note: The existence condition of laplace transformation of a function is sufficient but not necessary; following example establish it.

Example.

For example let $f(t) = \frac{1}{\sqrt{t}}$.

The function is not piecewise continuous for $[0, \infty)$

Now $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$= \int_0^\infty e^{-st} \frac{1}{\sqrt{t}}\,dt$,

Let $st = x$

or, $s\,dt = dx$

or, $dt = \frac{1}{s}\,dx$

t	0	∞
x	0	∞

So the integral becomes.

$$= \int_0^\infty e^{-x} \cdot \frac{1}{\sqrt{x/s}} \cdot \frac{1}{s}\,dx$$

$$= \frac{1}{\sqrt{s}} \int_0^\infty e^{-x} \cdot \frac{1}{\sqrt{x}}\,dx$$

$$= \frac{1}{\sqrt{s}} \int_0^\infty e^{-x} x^{-1/2}\,dx$$

$$= \frac{1}{\sqrt{s}} \int_0^\infty e^{-x} x^{1/2 - 1}\,dx$$

$$= \frac{1}{\sqrt{s}} \Gamma(1/2)$$

$= \sqrt{\pi/s}, \quad s > 0$ $\quad [\because \Gamma(1/2) = \sqrt{\pi}]$

This shows that $L(f(t))$ exists though it is not piece-wise continuous in $[0, \infty)$