B.Sc. II (Math (Hons))

Study Material

Paper – 4

Topic : Laplace Transformation

Material Sl. no – 5

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Laplace Transformation

Change of Scale.

If $L(f(t)) = F(s)$, then $L(f(at)) = \frac{1}{a} F\left(\frac{s}{a}\right)$

Proof

$$L(f(at)) = \int_0^\infty e^{-st} f(at)\, dt$$

Let $at = x$

$a\,dt = dx$

or, $dt = \frac{dx}{a}$

t	0	∞
x	0	∞

So the integral becomes.

$$\int_0^\infty e^{-sx/a} f(x)\, \frac{dx}{a}$$

$$= \frac{1}{a} \int_0^\infty e^{-s(x/a)} \cdot f(x)\, dx$$

$$= \frac{1}{a} \int_0^\infty e^{-\left(\frac{s}{a}\right)x} f(x)\, dx$$

$$= \cancel{\frac{1}{a} L(f(t))}$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right)$$

Multiplication by "t^n"

If $L\{f(t)\} = F(s)$, then

$$L(t^n f(t)) = (-1)^n \frac{d^n}{ds^n} F(s), \quad \forall n \in \mathbb{N}.$$

Proof: We prove this by mathematical Induction

We know that $F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$

$$\therefore \frac{d}{ds} F(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty \frac{\partial}{\partial s}(e^{-st} f(t))\, dt$$

$$= \int_0^\infty [f(t)\cdot(-t)\, e^{-st}]\, dt$$

$$= -\int_0^\infty e^{-st}\{t f(t)\}\, dt$$

$$= -L(t f(t))$$

$$\therefore L(t f(t)) = (-1)^1 \frac{d}{ds} F(s)$$

This shows that the theorem is true for $n = 1$.

If possible, let us assume that the theorem holds for $n = m$.

So, $L(t^m f(t)) = (-1)^m \frac{d^m}{ds^m} F(s)$

differentiating w.r.t s we get

$$\frac{d}{ds} L\{t^m f(t)\} = \frac{d}{ds}\left[(-1)^m \frac{d^m}{ds^m} F(s)\right]$$

or, $\frac{d}{ds}\int_0^\infty e^{-st} t^m f(t)\,dt = (-1)^m \cdot \frac{d^{m+1}}{ds^{m+1}} F(s)$

or, $\int_0^\infty \frac{\partial}{\partial s}\left[e^{-st} t^m f(t)\right] dt = (-1)^m \frac{d^{m+1}}{ds^{m+1}} F(s)$

or, $\int_0^\infty t^m f(t)(-t)\, e^{-st}\, dt = (-1)^m \frac{d^{m+1}}{ds^{m+1}} F(s)$

or, $(-1)\int_0^\infty t^{m+1} f(t)\, e^{-st}\, dt = (-1)^m \frac{d^{m+1}}{ds^{m+1}} F(s)$

or, $\int_0^\infty e^{-st}\, t^{m+1} f(t)\, dt = (-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}} F(s)$

or, $L(t^{m+1} f(t)) = (-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}} F(s)$

This shows that the theorem is true for $m+1$ if it is true for m.

therefore by principle of Mathematical induction the theorem is true every n, $n \in \mathbb{N}$.

Note.

Putting $n=1$ in the above theorem we get

$$L(t f(t)) = -\frac{d}{ds} F(s)$$

Examples.

Find $L(t^2 e^{2t})$

Soln. We know that $L(e^{2t}) = \frac{1}{s-2}$, $s > 2$

$$L(t^2 e^{2t}) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s-2}\right)$$

$$= \frac{d}{ds}\left(\frac{d}{ds}\left(\frac{1}{s-2}\right)\right)$$

$$= \frac{d}{ds}\left(-\frac{1}{(s-2)^2}\right)$$

$$= +2\frac{1}{(s-2)^3}$$

$$= \frac{2}{(s-2)^3}$$