Study Material.

B.Sc. I (Math Hons)

Paper - 1

Set Theory.

Material sl. no. 1

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Paper – 1

Topic: Set Theory

Partial Order Relation: A relation R on a set S is called a partial order relation on S if it satisfies the following conditions:

(i) $(a,a) \in R$ for all $a \in S$ (i.e. R is reflexive)

(ii) For all $a, b \in S$, if $(a,b) \in R$ and $(b,a) \in R$, then $a = b$ (i.e. R is antisymmetric)

(iii) For all $a, b, c \in S$, if $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$ (i.e. R is transative)

Thus we can say that a reflexive, antisymmetric and transative relation on a set S is called a partial order relation on S.

A partial order on a set is ~~has~~ usually denoted by '$\leq$'

Example

1. Let R be a relation on $\mathbb{N}$, defined by $R = \{(a,b) : a$ divides b in $\mathbb{N}\}$. Then R is a partial order relation.

Let $n \in \mathbb{N}$, since $a|a$ and so $(a,a) \in R$.
Thus R is reflexive.

Let (a,b) & $(b,a) \in R$. Then $a|b$ and $b|a$.
Now $a|b$ implies $b = ak_1$ for some +ve integer k_1,
and $b|a$ implies $a = bk_2$ for some +ve integer k_2.
Therefore $a = bk_2 = ak_1k_2$. So $k_1k_2 = 1$

Since k_1 and k_2 are positive integers, the only possibility is $k_1 = k_2 = 1$.

Consequently $a = b$.
Therefore R is ~~an~~ antisymmetric.

~~Let for transition~~
Let $(a,b), (b,c) \in R$.

Then $a \mid b$ & $b \mid c$
Therefore $b = am_1$ and $c = bm_2$ for some positive integer m_1 and m_2.

Now $c = bm_2 = am_1m_2$, since $m_1, m \in \mathbb{N}$.
so $m_1m_2 \in \mathbb{N}$.
Thus $a \mid c$. and consequently R is transative.

Combining all the cases we can say that R is a partial order relation.

(2) Let A be a set and $P(A)$ be the power set of A.
Let R be a relation on $P(A)$ defined by
$R = \{(A,B), A \subseteq B\}$. Then R is a partial order relation on $P(A)$.

Since $A \subseteq A$, $\forall A \in P(A)$, so $(A,A) \in R$ $\forall A \in P(A)$
This shows that R is reflexive.

Let $(A,B), (B,A) \in R$.
Then $A \subseteq B$ and $B \subseteq A$. and therefore $A = B$
This shows that R antisymmetric.

Let (A,B) & $(B,C) \in R$

Then $A \subseteq B$ & $B \subseteq C$ and therefore $A \subseteq C$.
So, $(A,C) \in R$
Hence R is transative.
Consequently R is a partial order relation.

Partially Ordered Set

A set S together with a partial order is called partially ordered set (poset).

If S is a partially ordered set with partial order $\leq$, then we write $(S, \leq)$.

- Let S be a poset and $a, b \in S$. If either $a \leq b$ or $b \leq a$, then we say that a and b are **comparable**.

Linearly Ordered Set

A partially ordered set $(S, \leq)$ is called a <u>linearly ordered set</u> or a <u>chain</u> if for all $x, y \in S$ either $x \leq y$ or $y \leq x$.

In other words we can say that a linearly ordered set is a poset in which any two elements are comparable.